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ON THE DYNAMIC PROPERTIES OF GRAVITATIONAL FIELDS*

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A theory which makes it possible to classify the points of Riemannian space is used, together with the corresponding associated reference systems for the gravitational field in the classical general theory of relativity, to obtain the expression for the field energy density in the form of a four-dimensional scalar (not a pseudoscalar) and energy-impulse field tensor as an energy-impulse tensor of second rank (not a pseudotensor). The results refer to all types of the empty Riemannian space as classified by A.Z. Petrov, Penrose and Newman.

1. The problem of determining the energy, impulse and internal stresses in the gravitational field has been apparent in its physical theory ever since the birth of the ideas of the general theory of relativity. Many authors have proposed, in direct contradiction to the basic principles of covariance of the laws of physics, and in particular to the representations of the energy as four-dimensional scalar characteristics of the individualized objects, various variants of the expressions for the pseudotensor of energy-impulse, basic expressions on a number of physically unsatisfactory solutions of the problems concerning the dynamic characteristics of the gravitational fields (all formulas proposed for the energy-impulse pseudotensors imply that the empty Minkowski space has, in the corresponding coordinates, variable energy, impulse and internal stresses). In this connection we must, before anything else, stress clearly that in introducing the concept of energy of the gravitational field, as well as in all remaining cases of investigating the continua, we must define an individual three-dimensional volume, i.e. a local field element and the corresponding characteristic time, and base our discussions, as in every other cases using the models and definitions of the energy, essentially on the variational equation for the first law of thermodynamics.

Introduction of the concept of individual points and of the corresponding coordinate systems for the individualised points reduces, in the Riemannian space, to introducing a time-like vector field of the unit vector $\mathbf{u}$ determined by the Riemannian space itself and admitting the possibility of treating it as a field of four-dimensional velocities

$$d\mathbf{r}/ds = \mathbf{u}$$
$$(ds^2 = g_{ij}dx^i dx^j;\ d\mathbf{r} = dx^i \partial_i = \mathbf{u}ds;\quad i, j = 1, 2, 3, 4)$$

The envelope lines of the vector field $\mathbf{u}$ represent the world lines of the points individualised by the integration constants ξ^α, $\alpha = 1, 2, 3$. The latter represent the Lagrange coordinates which appear, alternately, in the course of determining the laws of motion $x^i = x^i(\xi^1, \xi^2, \xi^3, \xi^4)$, ξ^4.

All dynamic properties of the Riemannian space and consequently of the gravitational field, can be considered as mechanical properties of the fluxes of points individualised by the Lagrange coordinates /1/. Below we shall restrict ourselves to considering the energy density and energy-impulse tensor density of the gravitational field in vacuum, i.e. in the four-dimensional volume of Riemannian space free of material mass, electromagnetic field and other fields.

In an earlier paper /2/ the problem in question was completely solved for a general type Riemannian space, i.e. for the case when the space belong to the first type T_1 as defined by Petrov /3,4/. The present work extends the results obtained earlier for the case of algebraically degenerate Riemannian spaces found within finite volumes. The theory demands that additional analysis be carried out, dictated in general by the presence or absence of a unique reference system associated with the Riemannian space which can be constructed in unique manner for the degenerate types of the Riemannian spaces. The results which follow can be related not only to the classical general theory of relativity, but also to numerous other generalized theories in which the physical space is modelled by a pseudo-Riemannian space.

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When we determine the vacuum in the Riemannian space, we mean by that the Ricci tensor components are equal to zero in vacuum: $R_{ij}=0$. This implies that the Riemannian tensor with components R_{ijkl} coincides in vacuum with the Weyl tensor with components W_{ijkl} according to the well known relation

$$R_{ijkl}=W_{ijkl}+{}^{1}/_{2}\,(g_{ik}R_{jl}+g_{jl}R_{ik}-g_{jk}R_{il}-g_{il}R_{jk})-{}^{1}/_{6}\,(g_{ik}g_{jl}-g_{il}g_{jk})\,R$$

Since a number of conclusions that follow are based on the properties of the Weyl tensor only, it follows that they can be used in examples also in the general cases when $R_{ij}\neq 0$ and the Weyl tensor is not equal to the Riemannian tensor. It must be remembered here that the Weyl tensor represents one of the main geometrical characteristics of the Riemannian space also in the general case.

The geometrical and dynamic arguments that follow depend on utilizing the algebraic properties at the points and the analytic properties in the small neighborhood of the points belonging to the Riemannian space of the set of the Weyl tensor components.

Let us recall, before anything else, the definitions and results given by Petrov /3,4/, Debever /5/, Sachs /6/, Newman and Penrose /7/ in their papers. We introduce at every point of the four-dimensional pseudo-Reimannian space a symmetric dynamic matrix $\mathbf{K}$ of sixth rank formed by the Weyl tensor components different, on the whole, from zero. The matrix $\mathbf{K}$ is formed from the tensor components W_{ijkl}, with the row and column indices shown below

$$\mathbf{K}=\begin{array}{c|c|c} & 14,\ 24,\ 34,\ 23,\ 31,\ 12 & kl \\ \hline 14 & & \\ 24 & & \\ 34 & & \\ 23 & & \\ 31 & & \\ 12 & & \\ \hline ij & & \end{array}=\begin{Vmatrix}\mathbf{M} & \mathbf{N}\\ \mathbf{N} & -\mathbf{M}\end{Vmatrix}$$

where $\mathbf{M}$ and $\mathbf{N}$ are two symmetric matrices of third rank.

As we know, the matrix $\mathbf{K}$ can be reduced locally at every point of the Riemannian space, with help of real coordinate transformation, to the canonical form in the corresponding tetrad ∂_i formed by the system of four orthonormed unit basis vectors $\partial_1, \partial_2, \partial_3, \partial_4$

$$(\partial_1\cdot\partial_1=\partial_2\cdot\partial_2=\partial_3\cdot\partial_3=-1;\ \partial_4\cdot\partial_4=+1 \text{ and } \partial_i\cdot\partial_j=0 \text{ for } i\neq j)$$

of which the vector ∂_4 can be regarded as time-like and assumed to be pointed in the direction of increasing characteristic time for the element $d\mathbf{r}=\partial_4 ds$. Thus, according to Petrov the vector field $\mathbf{u}=\partial_4$ is uniquely defined in the type T_1 space. In the general case we use the local, linear nonholonomic coordinate transformations to obtain globally the result that the canonical forms of the matrices $\mathbf{M}$ and $\mathbf{N}$ have the following form for the type T_1:

$$\mathbf{M}=\begin{Vmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\\ 0 & 0 & -\alpha_1-\alpha_2\end{Vmatrix},\quad \mathbf{N}=\begin{Vmatrix}\beta_1 & 0 & 0\\ 0 & \beta_2 & 0\\ 0 & 0 & -\beta_1-\beta_2\end{Vmatrix}$$

Here α_1, α_2 and β_1, β_2 are invariants of the Weyl tensor and we either have $\alpha_1\neq\alpha_2$ and $\beta_1\neq\beta_2$, or $\alpha_1\neq-\alpha_2/2,\ \beta_1\neq\beta_2/2,\ \alpha_2\neq-\alpha_1/2,\ \beta_2\neq\beta_1/2$. Therefore the diagonal terms in the matrices $\mathbf{M}$ and $\mathbf{N}$ differ from each other, the fact related to the absence of multiple roots in the corresponding secular equation. The uniqueness of the canonical matrix $\mathbf{K}$ and the corresponding basis vectors $\partial_4=\mathbf{u}$ and their envelopes implies, in general, the uniqueness of the associated reference system. Degenerate types result from the multiple roots, for which the canonical forms of the matrices $\mathbf{M}$ and $\mathbf{N}$ retain, in the case of N, D and O a special form uniquely defined for each type, while the corresponding Petrov tetrads and the vector $\mathbf{u}=\partial_4$ are not determined uniquely.

Every system of tetrads defined by means of the vector fields $\mathbf{u}=\partial_4$ as canonical for the matrices of the Weyl tensor components in the given Riemannian space, has a corresponding family of world lines which can be regarded as the families of lines associated with the Riemannian space which has the corresponding, uniquely introduced canonical matrix of the Weyl tensor components. According to Petrov, the canonical tetrads in the types T_1, T_2 and T_3 are determined uniquely at every point M of the Riemannian space, therefore in these cases the associated reference system obtained is unique and fully defined by the Weyl tensor. In the types N, D and O we find that for the fixed canonical forms of the matrix $\mathbf{K}$ the corresponding orthonormed tetrads ∂_i and the corresponding vectors $\partial_4=\mathbf{u}$ are not, according to Petrov, defined uniquely. In the type O the tetrads are arbitrary. In the case of the types N and D we obtain, for the possible vectors $\mathbf{u}$, at every point M of the Riemannian space, families which depend, respectively, on two or on a single scalar parameter. A complete three-dimensional cone of directions can be constructed at every point in question of the Riemannian space M. Out of these directions we can, in general, extract six directions determined by the isotropic vectors in terms of the basis vectors taken in the Petrov tetrads $\mathbf{Q}^{*}_{\alpha\pm} = \mathfrak{Э}_4 \pm \mathfrak{Э}_{\alpha} (\alpha = 1, 2, 3)$, and hence construct the corresponding six families of the isotropic world lines which can also be regarded as lines directed towards the future and "accompanying", in the universe, the given Reimannian space. These lines can however turn into each other when the basis is transformed.

Apart from the isotropic lines accompaning with the space, we can introduce, by definition, the principal isotropic directions at every point of the space, tangent to the isotropic vectors the components of which satisfy the equations(*)

$$Q_{[i}W_{j]kl[m}Q_{n]}Q^kQ^l = 0 \tag{1.1}$$

in which the vectors $\mathbf{Q} = Q^i\mathfrak{Э}_i$ satisfying the conditions of isotropy $g_{pq}Q^pQ^q = 0$ are determined with the accuracy of up to the constant multiplier. In general, we have four different directions in the nondegenerate cases of type T_1, therefore we obtain four isotropic associated lines in the form of envelopes of the principal isotropic vectors $\mathbf{Q}$. These lines can also pass into each other during the corresponding coordinate transformations. The directions $\mathbf{Q}^{*}_{\alpha\pm}$ are on the whole, different from those of the principal isotropic vectors $\mathbf{Q}$.

If some of the solutions of (1.1) merge, then the Riemannian space becomes algebraically degenerate. If only two solutions merge into one, then the R space will be of type T_2 by definition. If three solutions for $\mathbf{Q}$ in (1.1) merge at every point, and therefore three principal isotropic directions merge, then we have a type T_3 space. Merger of all four solutions yields a type N space. If the principal directions merge in pairs, separately but at the same time, we have a type D space, and finally, a type O Riemannian space obtain when all components of the Weyl tensor are zero. In all the types described above the canonical forms of the matrices $\mathbf{K}$ are known /4/. When the canonical form of the matrix $\mathbf{K}$ is kept invariant, then the transformations taking the system of orthonormed bases $\mathfrak{Э}_i$ into the system of orthonormed bases $\overline{\mathfrak{Э}}_i$, can only represent a Lorentz transformation of the form

$$\overline{\mathfrak{Э}}_i = L_i^{\ k}\mathfrak{Э}_k \tag{1.2}$$

Direct substitution can be used to show that in the types T_1, T_2 and T_3 the above transformation can only be an identity. Consequently the tetrads $\mathfrak{Э}_i$ and the vector field $\mathbf{u} = \mathfrak{Э}_4$ are determined in these types uniquely and the associated reference system, i.e. the system of the world lines enveloping the vectors $\mathbf{u}$ describe, together with the Weyl tensor invariants, a Riemannian space and all its properties at $R_{ij} = 0$, including the dynamic properties of the R spaces for the case of vacuum.

The Weyl tensor has only four independent algebraic invariants. We can take as these invariants, four functions appearing in the canonical matrices, in terms of which all the Weyl tensor components written in canonical matrices are expressed. From the known types of the matrices $\mathbf{K}$ for the Weyl tensor it follows that in the types T_3, N, and O the invariant components of the Weyl tensor are either known constants, which can be assumed in accordance with the canonical forms of the matrices $\mathbf{M}$ and $\mathbf{N}$ equal to unity, or they are zero. For this reason we can treat any functions of the Weyl tensor invariants in the present cases as constants, although the values of these constants may depend on the type and form of the functions of the invariants used.

In the types N, D, and O the tetrads $\mathfrak{Э}_i$ are, according to Petrov, noninvariant under the condition of invariant determination of the corresponding canonical type of the matrices $\mathbf{K}$, and hence of $\mathbf{M}$ and $\mathbf{N}$. As we know, in these cases groups of transformations of the orthonormed bases $\mathfrak{Э}_i, L_O, L_N$ and L_D exist, which leave invariant the matrices $\mathbf{M}$ and $\mathbf{N}$ defined according to Petrov. In particular, for the type O such a group of transformations coincides with the complete group of Lorentz transformations corresponding to the passage between two arbitrary, fixed time-like directions of the vector $\mathbf{u} = \mathfrak{Э}_4$. (In this case we find that when $R_{ij} = 0$ the space becomes a Minkowski space). Therefore L_O has three real parameters, L_N has two and L_D has one, the parameters changing the associated reference systems.

In the cases listed above the vector field $\mathbf{u} = \mathfrak{Э}_4$ in the corresponding tetrads is not determined uniquely, consequently the associated reference systems cannot be determined uniquely by the canonical form of the matrix $\mathbf{K}$ only. We shall however show that in these cases we

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*) A natural route for introducing the principal isotropic directions and equations (1.1) can be found in papers /5 - 11/.

find that the problem belongs to the inertial reference systems just as in the special theory of relativity where the part played by the reference systems geometrically associated with the Minkowski space, in determined nonuniquely. In the case O the family of the associated world lines represents a system of parallel, straight time-like lines.

The nonuniqueness encountered in the types O, N and D is essentially connected with the presence in these spaces of the sets of equivalent global reference systems representing the straightforward analogs of the inertial reference systems in the special theory of relativity. We have the following relationship for the canonical matrices in the case when the principal isotropic directions $\mathbf{Q}$ merge, and the vectors $\mathbf{Э}_1, \mathbf{Э}_2, \mathbf{Э}_3$ in any tetrad corresponding to the canonical matrix $\mathbf{K}$, are designated in prescribed manner:

$$\mathbf{Q}_1 = \mathbf{Q}^*_{1+} = Э_4 + Э_1 \tag{1.3}$$

For this reason the formulas $\mathbf{Э}_1(1,0,0,0), \mathbf{Э}_4\ (0,0,0,1)$ and $\mathbf{Q}_1\,(1,\ 0,\ 0,\ 1)$ for the components $Э_1, Э_4$ and $\mathbf{Q}_1$ taken in the Petrov tetrads, hold for any tetrad $Э_i$ obtained from any canonical tetrad by means of an admissible Lorentz transformation. In the case D where we have two merged principal isotropic directions $\mathbf{Q}_1$ and $\mathbf{Q}_2$ which single out a plane invariant element π, corresponding to the basis vectors $\mathbf{Э}_4, \mathbf{Э}_1$ in the canonical tetrads, we can assume that in addition to (1.3) for $\mathbf{Q}_1$ the following relation holds for $\mathbf{Q}_2$

$$\mathbf{Q}_2 = \mathbf{Q}^*_{1-} = Э_4 - Э_1 \tag{1.4}$$

The formulas (1.3) and (1.4) follows from the properties of solutions of (1.1) written for the canonical matrices in the degenerate types D /4/.

Let us consider the problem of constructing the associate reference systems for the degenerate types O, N and D of the pseudo-Riemannian spaces in which the unit vector $\mathbf{Э}_4$ corresponding to the Petrov canonical orthonormed tetrads is not defined uniquely. In this connection we recall the actual canonical forms of the matrices $\mathbf{K}$ and the corresponding canonical tetrads in the types O, N and D. In the type O we have $W_{ijkl} = 0$ in all coordinate systems, and therefore, generally speaking, in all tetrads. This also implies that all invariants of the Weyl tensor can be assumed, in the type O, to be equal to zero. As we know, in vacuum where the relation $R_{ij} = 0$ also holds, the corresponding pseudo-Riemannian space degenerates to a Minkowski space.

Using the arbitrariness of the canonical tetrads in the Minkowski space, we can consider, in this space, any reference systems depending on the choice of the vector fields $\mathbf{u}$. The determination of the latter requires, generally speaking, the introduction of functions expressing their laws of distribution, depending essentially not only on the geometrical nature of the Minkowski space. Clearly, vector fields $\mathbf{u}$ of such nature, connected not only with the spacial characteristics, can be investigated in any type of the Riemannian space and with various corresponding Weyl tensors /12 - 14/. However, the theory developed here considers the vector fields for $\mathbf{u}$, which can be determined at every point of the space algebraically, only in terms of the metric tensor components g_{ij} and Weyl tensor components W_{ijkl}.

In type O we have $W_{ijkl} = 0$ at all points of the space. We can therefore assume that every reference system associated with the Minkowski space is fully determined by separating a single arbitrary initial tetrad, since there are no reasons connected with the character of the Minkowski space to suggest that the tetrads at the neighboring points should vary in relation to the initial tetrad chosen. This implies that after the initial tetrad has been chosen, all tetrads at the neighboring points and, generally speaking, at all other points of the Minkowski space, should be the same. In this case the field of tangential unit vectors $\mathbf{u}$ can be constructed in the local, as well as the global manner, starting from any single given tetrad at an arbitrarily chosen point, by consecutive displacement of the vector $\mathbf{u}$ into all adjacent, infinitesimally near points. In this manner we find that the reference numbers associated with the Minkowski space represent arbitrary families of the time-like parallel straight lines. It is clear that such reference systems represent, in the special theory of relativity, the inertial reference systems which can be regarded as the principal characteristic geometrical singularities of the Minkowski space. It is also clear that the algorithm for separating the characteristic associated reference systems in type O out of the nonuniquely defined vector fields $\mathbf{u}$ taken from the Petrov tetrad, is obtained thanks to the absence of any influence from any additional geometrical parameters.

In type N the canonical type matrices $\mathbf{M}$ and $\mathbf{N}$ in $\mathbf{K}$ have the following form /4/ for the nonuniquely defined canonical Petrov tetrads at any point of the space:

$$\mathbf{M} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}, \quad \mathbf{N} = \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$$

From this we see that in any concrete solutions of type N all scalar invariants of the Weyl tensor represents the same scalar constant over all points of the space.

Let ∂_i denote certain, arbitrarily chosen orthonormed vectors for the Petrov tetrad, defined at every point of the space N. Then we know /6-11/ that at any point of the space the canonical form of the matrix $\mathbf{K}$ remains invariant for all solutions of type N, in the tetrad $\bar{\partial}_i$, defined by the transformation

$$\bar{\partial}_4 = \partial_4 + \frac{1}{2}(a^2+b^2)(\partial_4+\partial_1) + a\partial_2 + b\partial_3 \qquad (1.5)$$

$$\bar{\partial}_1 = \partial_1 - \frac{1}{2}(a^2+b^2)(\partial_4+\partial_1) - a\partial_2 - b\partial_3$$

$$\bar{\partial}_2 = \partial_2 + a(\partial_4+\partial_1), \quad \bar{\partial}_3 = \partial_3 + b(\partial_4+\partial_1)$$

In the above transformation a and b denote any real functions of the points of the space. The transformations (1.5) represent the Lorentz transformations at any fixed point of the space, for any two parameters a and b.

It can be confirmed that the following invariant relations hold under the transformations (1.5) for the isotropic vector $\mathbf{Q}$ at every canonical tetrad, at every point of the space:

$$\mathbf{Q} = \mathbf{u} + \partial_1 = \bar{\mathbf{u}} + \bar{\partial}_1 \qquad (1.6)$$

The components of the vector $\mathbf{Q}(1,0,0,1)$ in the nonuniquely determined Petrov tetrads are defined uniquely by the canonical matrix $\mathbf{K}$. The isotropic vector $\mathbf{Q}$ represents, in the type N, four principal isotropic directions merged together. The components of the isotropic vector $\mathbf{Q}$ can be obtained in any concrete solution of type N, at every point of the space and in any coordinate system, by means of algebraic operations. To obtain all admissible vector fields $\mathbf{u}$ we define, at every point of the space, in addition to the partly defined field of canonical tetrads ∂_i, another two arbitrary scalar functions a and b of the points of the space, entering the transformation formulas (1.5).

In connection with solving the problem of determination of the reference systems associated with the type N space, we note that every concrete type N space has, as compared with type O space, additional geometrically essential field of the known principal isotropic vectors $\mathbf{Q}$. Apart from this vector, there are no other significant parameters at different points of the space, and in particular in the canonical matrix $\mathbf{K}$ which remains the same at all points and in any type N space.

At every point of the space N the system of canonical tetrads and the set of the corresponding basis vectors are characterized by a completely unique principal isotropic vector $\mathbf{Q}$. Also, at every point of the space N we find, for the group of transformations (1.5) of canonical tetrads representing a subgroup of the Lorentz transformations, that the geometrical locus of the ends of the vector $\mathbf{u}$ forms a two-dimensional surface ε.

From the first formula of (1.5) we find that the equation of the surface ε representing the geometrical locus of the ends of the vectors $\mathbf{u}'(u^1, u^2, u^3, u^4)$, has the form

$$x^4 = 1 + {}^1/_2\,[(x^2)^2 + (x^3)^2], \quad x^1 = -{}^1/_2\,[(x^2)^2 + (x^3)^2]$$

in any canonical tetrad with local Cartesian coordinates x^1, x^2, x^3, x^4. Clearly, the points on the surface ε belong to the three-dimensional hyperplane $x^4 = 1 - x^1$. In every canonical tetrad the plane perpendicular to the vectors ∂_2 and ∂_3 contains the vector $\mathbf{Q} = \partial_4 + \partial_1$.

Let us consider any two points P and P', but infinitely close to each other, and let $\mathbf{Q}(P)$ and $\mathbf{Q}'(P')$ denote the corresponding to them merged principal isotropic vectors, with ∂_i and ∂_i' denoting any two corresponding systems of canonical tetrads. For the infinitely close points P and P' we can transform all tetrads and the vector $\mathbf{Q}'$, parallel to themselves, from the point P' to the point P and describe all vectors using a single fixed Cartesian coordinate system of the tetrad ∂_i at the point P. If $\mathbf{Q}(P) = \mathbf{Q}'(P')$ for any P', then it is clear that any two-dimensional sets ε and ε' are identical and will coincide exactly after being translated to the point P in the system of bases ∂_i. In this case all vectors $\partial_4 = \mathbf{u}$ depending on the parameters a and b in the transformation (1.5) and defining ε, can be assumed, as in type O, to be the same at every point of the space. However, a constraint will now apply, namely the requirement that the ends of these vectors must belong to the two-dimensional set ε. Thus the vectors u defined at every point of the space in terms of two parameters $a = \text{const}$, $b = \text{const}$ will, as in type O, define the associated reference systems for which the following obvious relation will hold:

$$\nabla_i u^i = 0 \qquad (1.7)$$

If $\mathbf{Q}(P) \neq \mathbf{Q}'(P')$, then transporting the canonical tetrads from the point P' to point P we obtain different tetrad families. As we know, transforming one orthonormed tetrad into another orthonormed tetrad can always be carried out with help of the Lorentz transformation. However, the transformation of one canonical tetrad at the point P' into another canonical tetrad at the point P does not, in general, represent a Lorentz transformation of the type (1.5), since under the transformation (1.5) we always have $\mathbf{Q} = \mathbf{Q}'$ which contradicts the initial assumption that $\mathbf{Q}(P) \neq \mathbf{Q}'(P')$. Therefore, although the two-dimensional surfaces ε and ε' are the same, they have different orientations. The surfaces ε at the point P and P' transported from P' into P may have common points at which the vectors $\mathbf{u}$ are the same, but in this case the vectors $\mathbf{\Im}_1, \mathbf{\Im}_2, \mathbf{\Im}_3$ and $\mathbf{\Im}_1', \mathbf{\Im}_2', \mathbf{\Im}_3'$ are different in accordance, in particular, with equality (1.6) when $\mathbf{Q} \neq \mathbf{Q}'$. In the general case the sets ε and ε' coincide for the points P and P' belonging to one and the same envelope of the merged principal isotropic vectors $\mathbf{Q}$ which, as we know /9/, are for type N the geodesic vectors in the Riemannian space when $R_{ij} = 0$.

Next we consider the problem of constructing the associated reference system based only on the geometrical characteristics of the space itself, expressed as the envelopes of the vectors $\mathbf{u}$ for $\mathbf{Q} \neq \mathbf{Q}'$. We recall that the components of the vector $\mathbf{u}$ in the corresponding canonical tetrad are $0, 0, 0, +1$ and can be determined in any coordinate system using the known nonholonomic coordinate transformation to the given canonical tetrad. Keeping the vector $\mathbf{Q}$ fixed we can introduce, in a unique manner, at every point of the type N space using the set of canonical tetrads defined by the transformations (1.5), a two-dimensional surface ε orientated in the same manner with respect to all canonical tetrads at every point of the space. The orientation of the vector $\mathbf{Q}$ governs only the orientation of the equivalent tetrads from the point of the transformations (1.5) and the orientation of the surface ε at the points of the space.

Every individual point of the corresponding surface ε has, amongst the collection of equivalent tetrads, a corresponding, uniquely defined tetrad and the corresponding vectors $\mathbf{u}$ and $\mathbf{\Im}_1$ satisfying the relation (1.6). It is clear that the individualization, or in other words, separation of a point of the surface ε is equivalent to the process of separating a canonical tetrad and the vectors $\mathbf{u}'$ and $\mathbf{\Im}_1'$, and can be reduced to that of fixing the parameters a and b taken from the particular local transformation of the type (1.6). If we place all points of the space with the known vectors $\mathbf{Q}$ in 1:1 correspondence with the points of the surface ε or with the values of the parameters a and b obtained from the local partial transformation (1.5), we obtain the vector fields $\mathbf{Q}$, $\mathbf{u}'$ and $\mathbf{\Im}_1$ connected by the relation (1.6). This kind of individualization depends on the initial tetrad entering the formulas (1.5). It is however remarkable that the intrinsic properties of the surface ε do not depend on the choice of $\mathbf{u}$ in the initial tetrad. The orientation of the surface ε is different at different points of the space, and is completely defined, mainly by the isotropic vector $\mathbf{Q}$.

We can use as the parameters fixing the individual points on differently orientated surfaces ε at various points of the space, other parameters connected functionally in 1:1 correspondence with a and b. It is essential that the general equation of the surface ε is the same at every point of the space, in any canonical tetrad, and a fixed point on the surface ε has the corresponding, well defined vectors $\mathbf{Q}'$ and $\mathbf{u}'$ and a canonical tetrad. It should however be remembered that when $\mathbf{Q} \neq \mathbf{Q}'$, the systems of tetrads are different at different points P and P' of the space, and so are the surfaces $\varepsilon, \varepsilon'$. The unit vectors $\mathbf{u}$ and $\mathbf{u}'$ have different orientations. We have shown above that the points of the surface ε can always be individualized in the initial tetrad at the given point using the values of the parameters a and b appearing in the transformation (1.5). When $\mathbf{Q} \neq \mathbf{Q}'$, the same individual points on ε and ε' will have, at different points of the space, the corresponding different tetrads and different vectors $\mathbf{u}'$.

To construct different associated reference systems in the type N spaces, it is natural and sufficient to assume that all points of the space have a single corresponding point on the surface ε, with different orientation at different points of the space. The canonical tetrads and vectors $\mathbf{u}'$ with fixed values of the parameters a and b corresponding to the individualized point on the surface ε, for world lines and canonical tetrads at all points of the space. The Lorentz transformation for the tetrads lying infinitely near each other is determined by the matrices $\gamma_{ijl} dx^l$ where $\gamma_{ijl} = -\gamma_{jil}$ are the Ricci symbols and dx^l are the infinitesimal displacement vector components

$$\mathbf{P}'\mathbf{P} = d\mathbf{r}\,(dx^1,\ dx^2,\ dx^3,\ dx^4)$$

In accordance with this we obtain a definite reference system in the whole space. Further, if we accept the natural assumption that all points of the surface ε are equivalent, then every pair of parameters a and b, or in other words every individual point ε, will have its associated reference system. Thus we find that in type N a continuous collection of the associated reference systems can be used, every system determined by the values of the individual parameters of the points ε, and in particular by the values of the parameters a and b.

For a given, specified type N space we find, that according to (1.6) every scalar term in the equation

$$\nabla_i u^i + \mathrm{div}_4 \partial_1 = \nabla_i Q^i \tag{1.8}$$

has a single identical value for all associated reference systems. In particular, there exists an invariant $\nabla_i u^i$, which can depend on the points of the space and represents a characteristics of the type N space in question. The characteristics, as a function of the points of the space, can be different in the different, particular, type N spaces. It is clear that if in some region the directions of the vectors $\mathbf{u}$ and $\mathbf{u}'$ are the same at all points P and P' lying close to each other, but the canonical tetrads can be different, then the following relations hold:

$$\nabla_i u^i = 0, \quad \mathrm{div}_4 \partial_1 = \nabla_i Q^i$$

In different type N spaces the relation connecting the scalars $\nabla_i u^i$ and $\mathrm{div}_4 \partial_1$ depends on the distribution of the isotropic vectors $\mathbf{Q}$.

The method of constructing the associated reference systems in the Riemannian space represents, in type N, a straightforward natural generalization of the method used earlier to introduce the associated reference systems in the other types. Indeed, e.g. in type T_1 the canonical tetrad and vector $\mathbf{u}$ are determined uniquely, therefore the associated reference system is also unique. In type N the vector $\mathbf{u}$ is not determined uniquely by the form of the canonical matrix $\mathbf{K}$, but its uniqueness is attained by fixing the point on the surface ε or, which amounts to the same thing, fixing directly the vector $\mathbf{u}'$ obtained from the group of transformations (1.5). Now, in the case of N we can construct in this manner many associated systems, remembering to use different points of the surface ε. The requirement that the fixed points lie on the surface ε, or in other words, that the ends of the vectors $\mathbf{u}'$ lie on the surface ε, presents a certain restriction absent from the type O.

We can express the invariant $\nabla_i u'^i$ in type N, just as in type T_1, in terms of the characteristics of the associated reference system. Indeed, denoting by ∂_i' and ∂_i the orthonormed bases for the associated reference system in question at the points P' and P lying infinitely close to each other for the canonical tetrads, we find that the corresponding Lorentz transformation has the form

$$\partial_i' = (\delta_i^j + \gamma^{j}{}_{\cdot il} y^l)\, \partial_j$$

where y^l denote the Cartesian coordinates of the point P' in the tetrad for P, and γ_{ijl} are the Ricci symbols.

We can write for the vector $\mathbf{u}'$ in the tetrad for the point P /2/

$$\mathbf{u}' = \partial_4' = (\delta_4^j + \gamma^{j}{}_{\cdot 4l} y^l)\partial_j$$

and this yields

$$\nabla_i u^i = \gamma^{i}{}_{\cdot 4i} \tag{1.9}$$

From this it follows that the quantity $\gamma^{i}{}_{\cdot 4i}$ is a scalar with a single and the same value at every fixed point, and in all associated reference systems introduced above.

Let us now turn to type D. According to Petrov, in type D the canonical form of the matrices $\mathbf{M}$ and $\mathbf{N}$ in the matrix $\mathbf{K}$, for the nonuniquely determined canonical tetrads, is as follows:

$$\mathbf{M} = \begin{Vmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -2\alpha \end{Vmatrix}, \quad \mathbf{N} = \begin{Vmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & -2\beta \end{Vmatrix} \tag{1.10}$$

The invariants α and β in the actual solutions may depend on the points of the space. The principal, isotropic pairwise merged directions $\mathbf{Q}_1$ and $\mathbf{Q}_2$ define the plane π. A plane π can be introduced at every point of the space, together with the definite unit vectors ∂_4 and ∂_1 mutually perpendicular and lying in the plane π, in a nonunique manner. The vectors can be considered together with the corresponding unit vectors ∂_2 and ∂_3 as an orthonormed systems of the Petrov basis vectors in which the matrix $\mathbf{K}$ mentioned above and the corresponding matrices $\mathbf{M}$ and $\mathbf{N}$ (1.10) are invariant.

The nonuniqueness of the canonical Petrov tetrads is connected with the presence of a group of Lorentz transformations depending on a single parameter v/c, transforming the tetrad ∂_i into the tetrads $\bar{\partial}_i$ and retaining the canonical matrix of the Weyl tensor components invariant in the type D. As we know /9/, the transformations have the following form under the corresponding numbering of the orthonormed basis vectors ∂_i and $\bar{\partial}_i$ (here c denotes speed of light):

$$\bar{\partial}_4 = \frac{\partial_4 + (v/c)\,\partial_1}{\sqrt{1-(v^2/c^2)}}, \quad \bar{\partial}_2 = \partial_2 \tag{1.11}$$
$$\bar{\partial}_1 = \frac{(v/c)\,\partial_4 + \partial_1}{\sqrt{1-(v^2/c^2)}}, \quad \bar{\partial}_3 = \partial_3; \quad -c < v < c$$

Every such Lorentz transformation corresponds in a rectilinear translational motion along the direction of ∂_1 with three-dimensional velocity v of the translational motion in the plane π of the system $\bar{\partial}_1, \bar{\partial}_4$ relative to the system ∂_1, ∂_4. The formulas (1.11) provide the transformation from the canonical orthonormed unit bases $\partial_1, \partial_2, \partial_3, \partial_4$ to the canonical orthonormed unit bases $\bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3, \bar{\partial}_4$.

Let us now introduce the isotropic vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ directed along the principal, merged isotropic directions and defined by the formulas

$$\mathbf{Q}_1 = \mathbf{u} + \partial_1, \quad \mathbf{Q}_2 = \mathbf{u} - \partial_1 \tag{1.12}$$

The vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are situated in the plane π, just as the vectors $\mathbf{u}$ and ∂_1. The plane π, the directions of the vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ and α, β in the matrix $\mathbf{K}$ are all determined uniquely by the components of the metric tensor, and correspondingly by the components of the Weyl tensor. In case of different canonical tetrads, the vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are defined by virtue of the transformations (1.11) nonuniquely, since from the formulas (1.11) and (1.12) follow

$$\bar{\mathbf{Q}}_1 = \bar{\mathbf{u}} + \bar{\partial}_1 = (\mathbf{u} + \partial_1)\,\lambda = \lambda \mathbf{Q}_1 \tag{1.13}$$
$$\bar{\mathbf{Q}}_2 = \bar{\mathbf{u}} - \bar{\partial}_1 = (\mathbf{u} - \partial_1)\,\frac{1}{\lambda} = \frac{1}{\lambda}\mathbf{Q}_2, \quad \lambda = \sqrt{\frac{1+v/c}{1-v/c}}$$

where the scalar parameter λ is, according to (1.11), equivalent to the parameter v/c and can assume various values ranging from zero to $+\infty$.

The scalar equation $\mathbf{Q}_1\mathbf{Q}_2 = \bar{\mathbf{Q}}_1\bar{\mathbf{Q}}_2 = 2$ follows from the formulas (1.12) and is satisfied identically at every point of the space. When $v/c = 0$, and hence when $\lambda = 1$, we have from (1.11), $\bar{\partial}_4 = \partial_4$ and $\bar{\partial}_1 = \partial_1$. Thus the transformation formulas (1.11) are connected with the initial vectors ∂_4 and ∂_1 where $\lambda = 1$.

Let us now consider the geometrical locus of the ends of the vectors $\bar{\mathbf{u}} = \bar{\partial}_4$ for the Petrov tetrads at a fixed point of the space. Since this system of tetrads, and hence of the corresponding vectors $\bar{\mathbf{u}}$ depends on a single parameter only, it follows that the geometrical locus which represents in type D a simplified example of the set ε (two-dimensional in type N), represents here a curve situated in the plane π. It is clear that the plane π is different at different points of the space D, but the curve remains the same in every known plane π.

It is easy to establish that the set ε represents, for type D, a hyperbola in plane π with known asymptotes corresponding to two merged isotropic directions. The hyperbola is situated so that the vector $\bar{\mathbf{u}}$ can be represented by any vector pointing towards the future between the isotropic directions determined by the vectors $\mathbf{Q}_1$ and $\mathbf{Q}_2$. At every point of the type D space the hyperbola, as well as the principal merged isotropic directions and the plane π containing them, can be shown directly, provided that the components of the metric tensor in type D are known in some arbitrarily chosen coordinate system. Using this coordinate system at an arbitrarily selected point P, we can define arbitrarily a time-like unit vector $\mathbf{u}_0$ lying in the plane π, pointing towards the future and situated between the merged isotropic directions $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Having conditionally fixed the vector $\mathbf{u}_0$ corresponding to the values $v/c = 0$ or $\lambda = 1$, we can determine the individualization of the points lying on the hyperbola by the values of λ, or by the corresponding values of v/c in the Lorentz transformations (1.11) preserving the canonical form of the matrix $\mathbf{K}$. We use the vector $\mathbf{u}_0$ specified above and the corresponding canonical tetrad T to construct analogous tetrads T' and vectors $\mathbf{u}_0'$ at every point P' of the given space D and thus obtain the corresponding reference system associated with this space. We can carry out such construction in the small in more detail, as follows. We postulate the planes π and π' and the corresponding hyperbola at two infinitely close points P and P'. In addition to the unit vector $\mathbf{u}_0$ in the plane π and canonical tetrad T with the basis vectors $\partial_i\,(\partial_4 = \mathbf{u}_0)$, we take at the point P' of the plane π' an arbitrary unit vector $\mathbf{u}_0{*}$ and the corresponding canonical tetrad T^* with basis vectors $\partial_i{*}\,(\partial_4{*} = \mathbf{u}_0{*})$. We transfer the tetrad T^* and its basis vectors $\partial_i{*}$ from the point P' to the infinitely near point P and denote by L_{ij} the elements of the matrices $\mathbf{L}$ of the infinitely small Lorentz transformation, the elements connecting the vectors $\partial_i{*}$ of tetrad T^* with the vectors ∂_i of tetrad T. We have

$$\partial_i{*} = L_i{}^k \partial_k = E_i{}^s \Pi_s{}^k \partial_k$$

where the matrix $\Pi_s{}^k$ determines a three-dimensional turn, and the transformation matrix $E_i{}^s$ determines a translational motion of the type (1.11). As we know, the matrix $\mathbf{L} = \| L_i{}^k \|$ can be written for the infinitely small Lorentz transformation in the form

$$L = \| \delta_i{}^k + \gamma^k_{\cdot il}\, dx^l \| = E\Pi = \| \Pi_s{}^k E_i{}^s \| = \| E_i{}^s \Pi_s{}^k \|$$

in which

$$E = \begin{Vmatrix} 1 & 0 & 0 & \gamma^4_{\cdot 1l}\, dx^l \\ 0 & 1 & 0 & \gamma^4_{\cdot 2l}\, dx^l \\ 0 & 0 & 1 & \gamma^4_{\cdot 3l}\, dx^l \\ \gamma^1_{\cdot 4l}\, dx^l & \gamma^2_{\cdot 4l}\, dx^l & \gamma^3_{\cdot 4l}\, dx^l & 1 \end{Vmatrix} \qquad \Pi = \begin{Vmatrix} 1 & \gamma^2_{\cdot 1l}\, dx^l & \gamma^3_{\cdot 1l}\, dx^l & 0 \\ \gamma^1_{\cdot 2l}\, dx^l & 1 & \gamma^3_{\cdot 2l}\, dx^l & 0 \\ \gamma^1_{\cdot 3l}\, dx^l & \gamma^2_{\cdot 3l}\, dx^l & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix} \tag{1.14}$$

where the vector $\mathbf{PP}' = d\mathbf{r} = dx^l \mathbf{Э}_l = dx^k \mathbf{Э}_k{}^*$, $\gamma^k_{\cdot il}$ are Ricci symbols, and $\gamma_{kil} = -\gamma_{ikl}$.

The matrix Π determines an infinitely small, purely spatial turn which superimposes the planes π and π' and the corresponding hyperbolas, and the matrix E determines the relative translational motion of the bases $\mathbf{Э}_1, \mathbf{Э}_4 = \mathbf{u}_0$ and $\mathbf{Э}_1{}^*, \mathbf{Э}_4{}^* = \mathbf{u}^*$ along the hyperbola and depends only on the choice of $\mathbf{u}_0$ and $\mathbf{u}^*$.

Let us denote by $\| E_j{}^{*i} \| = E^*$ the matrix of the infinitely small Lorentz transformation in the plane π', inverse to the transformation E: $E_j{}^{*i} E_i{}^s = \delta_j{}^s$. If we now write

$$\mathbf{Э}_j{}' = E_j{}^{*i} \overset{*}{\mathbf{Э}}_i = E^*{}_j{}^i E_i{}^s \Pi_s{}^k \mathbf{Э}_k = \Pi_j{}^k \mathbf{Э}_k \tag{1.15}$$

then, irrespective of the choice of the initial vector $\mathbf{u}^*$ and the bases $\mathbf{Э}_i{}^*$ in the plane π', we can regard the canonical tetrad $\mathbf{Э}_i{}'$ and the vector $\mathbf{Э}_4{}' = \mathbf{u}'$ taken at the point P' as corresponding to the tetrad T and vector $\mathbf{u}_0$ at the point P. The tetrads T and T' are connected by an infinitely small turn under which the planes π and π' superimpose. The turns depend only on the vector $d\mathbf{r}$ and are independent of the choice of the initial vector $\mathbf{u}_0$ and consequently, of the values of the parameters v/c or λ individualizing the points on the hyperbolas in the planes π and π'. By virtue of (1.14) and (1.15) we find that the relation $u^i = u'^i$ holds for the infinitely close points P and P' in the corresponding canonical tetrads. If follows therefore that in a small region near the point P the components u^i taken in the neighboring canonical tetrads are the same to within the terms of higher order of smallness, and

$$\partial u^i / \partial x^k = 0, \ \operatorname{div}_4 \mathbf{u} = \nabla_i u^i = 0 \tag{1.16}$$

It is clear that the relations (1.16) hold at any point P of the space D and for every associated reference system constructed from any point P with any initial unit vector $\mathbf{u}_0$ of prescribed type.

In the tetrads differing from each other at the points P and P' we have, in the small, according to (1.15), $du^i/ds = 0$ where ds is an element in the metric form, with the accuracy of up to and including the first order infinitesimals in dx^l. However, the field of unit vectors $\mathbf{u}$ constructed step by step using the proposed method, with help of the canonical tetrads and with the nonlinear terms taken into account, in a finite region of the space D, is not obtained by parallel translation of the initial vector $\mathbf{u}_0$ selected at the some point P. In the small we have the same u^i in the neighboring tetrads, therefore $du^i/ds = 0$. In the general case however, we find that at different points of one and the same world line we have, for certain associated reference system, the acceleration $d\mathbf{u}/ds \neq 0$. This can easily be established when the components of the metric tensor of the given solution of type D are known. In this connection, it is evident that the corresponding world lines of the associated systems on which the accelerations are not zero, are not geodesic.

2. Let also inspect certain general conclusions and various representations of the metric form connected in any manner with the given field of four-dimensional velocities $\mathbf{u}$ and the associated coordinate systems. In the global form the metric form for the nonisotropic vectors can be written, in the general case, in the form

$$ds^2 = g_{44}(\eta^i)(d\eta^4)^2 + 2g_{\alpha 4}(\eta^i)d\eta^\alpha d\eta^4 + g_{\alpha\beta}d\eta^\alpha d\eta^\beta \tag{2.1}$$

or

$$ds^2 = c^2 dt^2 + 2\hat{g}_{\alpha 4} d\xi^\alpha dt + \hat{g}_{\alpha\beta} d\xi^\alpha d\xi^\beta \tag{2.2}$$

Here η^4, η^α and t, ξ^α, $\alpha = 1, 2, 3$ denote the Lagrangian coordinates connected by a transformation of the form

$$t = f(\eta^\alpha, \eta^4),\ \xi^\beta = \Phi^\beta(\eta^1, \eta^2, \eta^3);\ \alpha, \beta = 1, 2, 3 \tag{2.3}$$

It is evident that the formulas (2.1) and (2.2) correspond to the same family of world lines, determined by the same given unit vector field $\mathbf{u}$. It is easy to see that when ξ^α and η^α are constant, then the differential dt at the global time variable is equal to the increment in the characteristic time along every world line. The choice of the functions $f(\eta^\alpha, \eta^4)$ also determines the start of counting the characteristic time on every world line, and the functions $\Phi^\beta(\eta^1, \eta^2, \eta^3)$ can be arbitrary. In the general case the components g_{ij}, $i, j = 1, 2, 3, 4$, depend on η^1, η^2, η^3 and η^4 and the components $g_{\hat{\alpha 4}}$, $g_{\hat{\alpha\beta}}$ on all ξ^i. The family of the associated world coordinate lines $\xi^\alpha = \text{const}$ or $\eta^\alpha = \text{const}$ can be obtained, by virtue of the dependence of the coordinates η^i on t, from (2.1) and (2.2), using the formula (2.3).

We shall call the motion of a perfect medium with velocity $\mathbf{u}$ stationary, if all components of the metric tensor g_{ij} depend, in some coordinate system η^i, only on η^1, η^2 and η^3 and hence not on the time coordinate η^4. The corresponding form of the metric (2.1) and the corresponding motion may suggest the presence of stationarity, but in the form of the metric given by (2.2) for the same motion the components $g_{\hat{\alpha 4}}$ and $g_{\hat{\alpha\beta}}$ may depend on ξ^α and t.

In the general case it is impossible to make all components $g_{\hat{\alpha 4}}$ vanish when using the transformations (2.3) which preserve the reference system. Indeed, the four-dimensional velocity $\mathbf{u}$ and acceleration vectors $\mathbf{a} = d\mathbf{u}/ds$ represent the invariant geometrical characteristics of the given associated world lines. On the other hand, in a coordinate system corresponding to the form (2.2) of the metric and in a local inertial characteristic coordinate system with the orthonormed basis vectors $\bar{\mathbf{\partial}}_\alpha$, $\bar{\mathbf{\partial}}_4$, we have at any point M $\bar{\mathbf{u}} = \bar{u}_\alpha\bar{\mathbf{\partial}}^\alpha + \bar{u}_4\bar{\mathbf{\partial}}^4$ where $\bar{u}_\alpha = \bar{u}^k g_{\hat{\alpha k}} = g_{\hat{\alpha 4}}$, since

$$\bar{u}^4 = 1,\ \bar{u}^\beta = 0 \tag{2.4}$$

and by virtue of the inertial character of the tetrad $\bar{\mathbf{\partial}}_i$ we obtain

$$\frac{d\bar{\mathbf{u}}}{ds} = \frac{d\bar{u}_\alpha}{ds}\bar{\mathbf{\partial}}^\alpha = \frac{\partial g_{\hat{\alpha 4}}(t, \xi^\alpha)}{c\,\partial t}\bar{\mathbf{\partial}}^\alpha \tag{2.5}$$

Therefore, if the world lines are non-geodesic, then the acceleration $\mathbf{a} \neq 0$ and hence, in accordance with (2.5), $g_{\hat{\alpha 4}} \neq 0$ is mandatory in the associated reference system. If the components $g_{\hat{\alpha 4}}$ in (2.2) do not depend on the global time t, then the world lines are geodesic. If $g_{ij}(\eta^\alpha)$ depends only on η^α then we have stationarity, but the world lines $\eta^\alpha = \text{const}$ are, generally speaking, not geodesic.

If the world lines are geodesic, then $dg_{\hat{\alpha 4}}/dt = 0$ but in this case we can also make the components $g_{\hat{\alpha 4}}$ vanish by means of a transformation of the type (2.3), provided that the following condition of integrability holds:

$$\frac{\partial g_{\hat{\alpha 4}}}{\partial \xi^\beta} - \frac{\partial g_{\hat{\beta 4}}}{\partial \xi^\alpha} = 0 \tag{2.6}$$

or, in other words, provided that the corresponding velocity field with components u_α is irrotational. Arranging the velocity field $\mathbf{u}$, we can determine and compute at every point of the space, various mechanical characteristics of the flux of the corresponding perfect medium corresponding to the velocity field $\mathbf{u}$ at every point of the space.

Next we shall give a kinematic interpretation of the invariant $\nabla_i u^i$. In the locally inertial tetrad introduced above and corresponding to the formula (2.4), we have

$$ds^2 = dx^{4^2} - dx^{1^2} - dx^{2^2} - dx^{3^2},\quad dx^4 = c\,dt \tag{2.7}$$
$$\nabla_i u^i = \frac{\partial u^4}{\partial x^4} + \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3}$$

Let us now introudce the three-dimensional velocity $\mathbf{v}$ of the points $\mathbf{v} = v^\alpha\bar{\mathbf{\partial}}_\alpha$ in the three-dimensional space using the local characteristic coordinate system introduced at the point M. By definition, we have at M, $u_\alpha = 0$ and $v_\alpha = 0$. In the adjacent, infinietely close points M' we have, generally speaking, $u_\alpha \neq 0$ and $v_\alpha \neq 0$. We recall the known formulas

$$\frac{dt}{ds} = u^4 = \frac{1}{c\sqrt{1 - v^2/c^2}},\quad \frac{dx^\alpha}{ds} = u^\alpha = \frac{v^\alpha}{c\sqrt{1 - v^2/c^2}} \tag{2.8}$$

where c is the speed of light. Using (2.8) we obtain from (2.7) at the point M

$$\nabla_i u^i\, dV_4 = \frac{1}{c}\operatorname{div}_{x\alpha} \mathbf{v}\, dV_4 = \frac{dV_3' - dV_3}{dV_3 c\, dt} dV_4 = dV_3' - dV_3 \tag{2.9}$$

since we can assume that $dV_3 ds = dV_4 = dV_3 cdt$ where $ds = cdt$ and dt is the characteristic time increment on the world line passing through M. Here dV_3 denotes an infinitesimal element of three-dimensional volume orthogonal to the world line at the point M, while dV_3' is the same deformed "liquid volume" on the same world line, displaced over the characteristic time dt. In general, we can use for dV_3 a formula of the type

$$dV_3 = \left| g_{\alpha\beta} - \frac{g_{\alpha 4} g_{\beta 4}}{g_{44}} \right|^{1/2} d\eta^1 d\eta^2 d\eta^3 \tag{2.10}$$

where a known determinant appears under the square root sign. A corresponding formula holds for dV_3'.

If the motion with velocities $\mathbf{u}$ is steady, then all g_{ij} are independent of η^4 and $dV_3' = dV_3$. Consequently, if the motion of the perfect medium along the world line is stable, then from (2.9) we obtain

$$\nabla_i u^i = 0 \tag{2.11}$$

However, in this case, just as in the general case, after transforming (2.1) to (2.2) we find that the components $g_{4\hat{\alpha}}$ and $g_{\hat{\alpha}\hat{\beta}}$ will depend not only on ξ^α, but also on $\xi^4 = t$, and for this reason we shall have in the stable motions

$$u^{\wedge k} \nabla_{\hat{k}} u^{\wedge i} \neq 0, \quad d\mathbf{u}/ds = \mathbf{a} \neq 0$$

Equation (2.11) holds for the stable motions, but the world lines are, in general, no longer geodesic. Clearly, the relation $\nabla_i u^i = 0$ holds for the stable motion in any Petrov type spaces, for any associate reference systems at every point of the space.

3. All possible type D solutions in vacuum are known for the metric tensor components g_{ij} and have been published in /15/. The general symmetry properties of the type D space in vacuum imply that a coordinate system in which the components g_{ij} depend on two coordinates only, can always be found. If g_{ij} depend only on x^1 and x^2, then every associated coordinate system and the velocity field $\mathbf{u}$ corresponding to this reference system, will be stable. This implies that to determine the stable velocity field $\mathbf{u}$ in type D it is sufficient to reduce, with help of the symmetry properties, the metric (2.1) to the form in which the components g_{ij} are independent of x^4.

Above we constructed a series of the associated reference systems from the type D spaces, depending on a single parameter. It is clear that the relation $\nabla_i u^i = 0$ holds for all world lines in these systems in any coordinates. This implies that in type D solutions such as the Schwartzchild and Kerry solutions outside the gravitational sphere where the motion is steady state, we have $\nabla_i u^i = 0$ in the associated reference system constructed. In the types T_1, T_2, T_3 the vector field $\mathbf{u}$ and the associated reference systems are determined uniquely /3, 4,16/, while in the types D, N and O they are not, but in all cases the quantity $\nabla_i u^i$ is determined uniquely at every point of the space. The formula (2.9) provides a simple geometrical interpretation for the variation in a substantial three-dimensional volume from the point of view of a flux of individualized points moving with four-dimensional velosity $\mathbf{u}$. This is found to be an invariant feature of the Riemannian space, and a geometrically invariant property of the gravitational fields in vacuum in the general theory of relativity. The uniqueness of the determination of $\nabla_i u^i$ implies the uniqueness of variation in $dV_3' - dV_3$ along the associated world lines introduced above.

All previous discussions concerned the intrinsic geometrical properties of the Riemannian space connected with the Weyl tensor. The associated reference systems introduced are determined using a Weyl tensor of fourth rank, possessing the known symmetry properties and known types of canonical matrices in various degenerate cases. Thus we have developed above a purely mathematical geometrical theory for a four-dimensional, pseudo-Riemannian space, and introduced invariant reference systems determined by the Weyl tensor. The results obtained above have inherent mathematical interest and do not depend on any assumptions or postulates of a physical nature. However, the reference systems obtained can be regarded as a generalization of the inertial reference systems for the Minkowski space, and their characteristic features can be used as a basis for the physical assumptions concerning the determination of the energy and energy-impulse tensor of the gravitational fields considered in the Riemannian space in the case when the Ricci tensor is zero, i.e. when $R_{ij} = 0$.

4. In order to arrive at conclusions of physical nature it is expedient to begin with the basic integral variational equation representing, for an infinitesival element of space volume, medium and the corresponding fields, a direct consequence of the variational formulation in the small of the first and second law of thermodynamics /17 - 22/

$$\delta \int_{V_4} \Lambda \, dV_4 + \delta W^* + \delta W = 0 \qquad (4.1)$$

The Lagrangian Λ represents a scalar function which can be regarded as the total specific energy of the system taken with a minus sign, then term δW^* is governed by the presence of irreversible effects and external interactions, and the virtual functional δW obtained from (4.1) represents a surface integral taken over the boundary surface Σ enclosing an arbitrary region of volume V_4 in the course of integrating the continuous characteristic functions of the physical phenomena.

We find that in the problems of determining the energy and energy-impulse tensor of the gravitational field, insufficient attention is usually given in the general theory of relativity to the following aspects.

1) to the physical meaning itself of the variational equation (4.1) which is used, as a rule, in the postulated formulations of particular type:

2) to the analysis of possible expressions for the density of the Lagrangian when the Euler equations are fixed;

3) to the relation connecting the variational equation with the energy equation for the infinitesimal individualized objects and to the problems of individualization of the elements belonging to the system in question generally;

4) to the physical meaning of the energy-impulse tensor as a physical characteristics appearing in the equation of energy for the individualized infinitesimal volumes;

5) to the meaning of the divergent term in the expression for the Lagrangian not affecting the Euler equation, but affecting the expression for the energy-impulse tensor.

As we know, the Euler equations derived from (4.1) remain unchanged if an additive term of the form $-\nabla_i \Omega^i$ is included in Λ, or in other words, if Λ is replaced by

$$\Lambda' = \Lambda - \nabla_i \Omega^i \qquad (4.2)$$

where Ω^i $(i = 1, 2, 3, 4)$ are certain functions of the coordinates for which the formally and mathematically constructed expression $\nabla_i \Omega^i$ may, in general, not be a scalar. The sufficient condition for it to be a scalar is, that Ω^i represent the components of some vector $\Omega = \Omega^i \partial_i$. If $\nabla_i \Omega^i \neq 0$, then in physical terms it means that additional energy density can be introduced to the system in question. In the general theory of relativity this can be introduced in the form of a fraction corresponding to the energy field density and representing a physical characteristics of the four-dimensional Riemannian space modelling the physical space in nature. The fundamental physical concept of the energy related to the covariance of the physical laws, demands that the quantities Λ and $\nabla_i \Omega^i$ must be four-dimensional scalars.

It is improtant that the scalar density of the fraction $\nabla_i \Omega^i$ of energy should be representable by its geometrical properties, and it can be used in various applications of the general theory of relativity, on the whole independently of Λ determining the Euler equations and containing the energy of matter and electromagentic field, and other terms, the applications governed by the geometry of the Riemannian space and the properties which become zero in vacuum. Thus e.g. in the classical general theory of relativity the following formula for Λ is often used:

$$\Lambda = -\frac{R}{2\varkappa} - U_m$$

where R denotes the total curvature of the four-dimensional Riemannian space, U_m is the specific energy of matter referred to the four-dimensional volume, and $\varkappa$ is the gravitational constant. By virtue of the Euler equation we find that in vacuum, i.e. when $U_m = 0$, $R = 0$ and $\Lambda = 0$. Since the equation (4.1) represents, at $\delta W^* \neq 0$, a variational formulation of the first law of thermodynamics in which, as we already said, Λ is specific local energy, the equality $\Lambda = 0$ when $\delta W^* = 0$, contradicts the fundamental physical proposition that the gravitational fields have, and can transmit energy by gravitational waves. Addition of an invariantly defined term of the form $-\nabla_i \Omega^i$ to Λ in (4.1) restores in natural manner the physical sense of the first law of thermodynamics as applied to a gravitational field in vacuum. Its presence determines the additional energy $-\nabla_i \Omega^i dV_3$ and governs the appearance of the extra term δW_Ω, carrying an additional terms in the original expression for the energy-impusle tensor given in terms of Λ and δW^* in the basic equation (4.1). The basic Euler equations remain in this case completely unchanged.

The presence of an additional term

$$\int_{V_4} \nabla_i \Omega^i \, dV_4$$

in the first integral in (4.1) does not affect the Euler equations, since the integral can be transformed into a surface integral over Σ and all variations on Σ can be assumed, when deriving the Euler equations, to be equal to zero. The vector $\Omega = \Omega^i \partial_i$ can in general depend on a number of additional intrinsic parameters affecting the energy flux through Σ. The corresponding intrinsic parameters can be used in the physical theories in which certain complex properties are assigned to the vacuum. Using the equation (4.1), we can write

$$-\delta \int_{V_4} \nabla_i \Omega^i \, dV_4 + \delta W_\Omega = 0$$

From this we find that, provided that Ω^i depends on the Lagrangian coordinates ξ^i only, $\delta\Omega^i = 0 = \partial\Omega^i + \delta x^j \nabla_j \Omega^j$ and the following formula holds:

$$\delta W_\Omega = \int_\Sigma P_j{}^k \delta x^j n_k \, d\sigma = \delta \int_{V_4} \nabla_i \Omega^i \, dV_4$$

In this case the determination of δW_Ω is reduced to determination of $P_j{}^k$, i.e. the components of a tensor, provided that $\nabla_i \Omega^i$ is a scalar. After carrying out the variations and corresponding transformations we obtain, in the present case, in account of δW_Ω (see /17/), the following expression for the added part of the energy-impulse tensor component $P_j{}^k$:

$$P_j{}^k = \nabla_i \Omega^i \delta_j{}^k - \nabla_j \Omega^k \tag{4.3}$$

In the general case it is easy to confirm that

$$\nabla_k P_j{}^k = (\nabla_j \nabla_k - \nabla_k \nabla_j) \Omega^k = R^k_{\cdot mjk} \Omega^m = -R_{mj} \Omega^m \tag{4.4}$$

In vacuum where $R_{mj} = 0$, we obtain $\nabla_k P_j{}^k = 0$ and from the equality (4.3) follows

$$P_4{}^4 = \nabla_\alpha \Omega^\alpha \quad (\alpha = 1, 2, 3) \tag{4.5}$$

Setting $\Omega = I\mathbf{u}$, where I is a certain function of the Weyl tensor invariants, we obtain in the reference system associated with the vector $\mathbf{u}$

$$\nabla_i \Omega^i = \nabla_i (I u^i) = \frac{dI}{ds} + I \nabla_i u^i, \quad \nabla_4 \Omega^4 = \frac{dI}{ds} + I \nabla_4 u^4 \tag{4.6}$$

Therefore from (4.5) and (4.6) we find that in the reference system associated with the vector $\mathbf{u}$ and along the world lines, the following relations hold:

$$P_4{}^4 = \nabla_i (I u^i) = I \nabla_i u^i = I \nabla_\alpha u^\alpha, \; \partial u^4 / \partial t = 0 \tag{4.7}$$

From (4.3) for $P_j{}^k$ and from the supplementary assumption that in the associated coordinate system the following relation must hold together with (4.7) for the individualized infinitesimal volumes contracting into a point:

$$P_4{}^4 = \nabla_i \Omega^i = \nabla_i I u^i = \frac{dI}{ds} + I \nabla_i u^i \tag{4.8}$$

Relations (4.7) and (4.8) agree, provided that the following equation holds along every world line:

$$dI/ds = 0 \tag{4.9}$$

In this case we obtain

$$P_4{}^4 dV_4 = I \nabla_\alpha u^\alpha dV_4 = I \, (dV_3' - dV_3) \tag{4.10}$$

since $du^4/d\xi^4 = 0$. Thus from (4.9) it follows that

$$I\,(\xi^1, \xi^2, \xi^3) = I\,(\eta^1, \eta^2, \eta^3) \tag{4.11}$$

Clearly, the relation (4.11) will hold in any Lagrangian coordinate system provided that the metric tensor components are independent of η^4. Otherwise, (4.9) implies that we must take, as the invariant I, a constant scalar in every of the types T_1, T_2 and D as well as in T_3, N and O. We have shown above that in the types T_3, N and O the invariant I cannot differ from a constant. From (4.8) and (4.9) it follows that the specific energy of the gravitational field in vacuum per unit four-dimensional volume, can be obtained from the formula

$$P_4{}^4 = \varepsilon = \nabla_i I u^i = I \nabla_\alpha u^\alpha \tag{4.12}$$

where I is a certain constant not only with respect to the time coordinate, but, generally speaking, also to the spatial coordinates. From (4.3) and (4.12) follows a formula for the energy-impulse tensor of the gravitational field in vacuum

$$P_j{}^k = I\,(\nabla_i u^i \delta_j{}^k - \nabla_j u^k) = I\nabla_i\,(u^i \delta_j{}^k - \delta_j{}^i u^k)$$

The above results can also be used in the macroscopic theories concerning the energy-impulse tensor in the presence of matter and electromagnetic field.

As a result of the mathematically and physically correct theory for the Riemannian space developed above, we introduce the associated reference systems and formulas determined by the space itself, for the energy-impulse tensor components in all possible examples of the gravitational fields in vacuum. The results obtained make it possible to write the conditions at the strong discontinuities which may appear within the gravitational fields, and can be used in formulating the boundary conditions.

Apart from the canonical associated systems determined only by the algebraic properties of the Weyl tensor components in each type, we introduce many other reference systems corresponding to other vector fields **u**. However, the determination of every different vector field **u** is connected either with use of the higher order derivatives in x^i of the metric tensor components, or with the use of some parameters the nature of which cannot be determined directly and exclusively by the local properties of the Weyl tensor for the Riemannian space. Such supplementary parameters can result in substantial inequalities $\nabla_i u^i \neq 0$ for the associated reference systems in the space of type D and Minkowski space, which cannot be regarded, under any circumstance, and the characteristic feature of those spaces.

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